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The algebra of bounded linear operators on $\ell_p \oplus \ell_q$ has infinitely many closed ideals

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Dedicated to the memory of Ted Odell

Abstract. We prove that in the reflexive range $1 < p < q < \infty$, the algebra $\mathcal{L}(\ell_p \oplus \ell_q)$ of all bounded linear operators on $\ell_p \oplus \ell_q$ has infinitely many closed ideals. This solves a problem raised by A. Pietsch [4, Problem 5.3.3] in his book 'Operator ideals'.

1. Introduction

The classification of the closed ideals of the algebra $\mathcal{L}(\ell_p \oplus \ell_q)$ of bounded linear operators on $\ell_p \oplus \ell_q$ has a long history. There were several results proved in the 1970s, and the reader is referred to the book by A. Pietsch [4, Chapter 5] for details. In particular, a result of P. Volkmann [9] (see also [4, Theorem 5.3.2]) asserts that for $1 \leq p < q < \infty$ there are exactly two maximal ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$. These are the closures of the ideals of operators factoring through ℓ_p and ℓ_q, respectively. In [4, Theorem 5.3.2] a one-to-one correspondence is established between the set of all other closed, proper ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$ and the set of all closed ideals of $\mathcal{L}(\ell_p, \ell_q)$. Here an ideal of $\mathcal{L}(\ell_p, \ell_q)$ is a subspace $\mathcal{J}$ of $\mathcal{L}(\ell_p, \ell_q)$ with the property that $ATB \in \mathcal{J}$ whenever $A \in \mathcal{L}(\ell_q)$, $T \in \mathcal{J}$ and $B \in \mathcal{L}(\ell_p)$. Pietsch raises the following problem.

Problem ([4, Problem 5.3.3]). For $1 \leq p < q < \infty$ does $\mathcal{L}(\ell_p, \ell_q)$ have infinitely many closed ideals?

Since we are dealing with Banach spaces with bases, it is clear that the compact operators $\mathcal{K} = \mathcal{K}(\ell_p, \ell_q)$ is the smallest non-trivial (i.e., non-zero) closed ideal. Since the formal inclusion map $I_{p,q}: \ell_p \to \ell_q$ is not compact, $\mathcal{K}$ is a proper ideal. For anyone well versed in basis

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techniques, it is a not too difficult exercise that every operator in $\mathcal{L}(\ell_p, \ell_q)$ is strictly singular, and that every non-compact operator factors $I_{p,q}$. It follows that

$$\{0\} \subsetneq \mathcal{K} \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{S} = \mathcal{L}(\ell_p, \ell_q),$$

where $\mathcal{J}^{I_{p,q}}$ is the closure of the ideal of operators factoring through $I_{p,q}$, $\mathcal{S} = \mathcal{S}(\ell_p, \ell_q)$ is the ideal of strictly singular operators, and moreover any other closed ideal of $\mathcal{L}(\ell_p, \ell_q)$ must contain $\mathcal{J}^{I_{p,q}}$. It is, however, not obvious that $\mathcal{J}^{I_{p,q}}$ is proper. This was shown for $1 < p < q < \infty$ by V. D. Milman [3] who first proved that $I_{p,q}$ is finitely strictly singular, and then exhibited an operator in $\mathcal{L}(\ell_p, \ell_q)$ that is not finitely strictly singular. (Definitions will be given in Section 2 below.) The next significant result was proved by B. Sari, N. Tomczak-Jaegermann, V. G. Troitsky and the first named author. In [7] they showed that for $1 < p < 2 < q < \infty$, the ideal $\mathcal{FS}$ of finitely strictly singular operators and the ideal $\mathcal{J}^{\ell_2}$ generated by operators factoring through ℓ_2 are proper, distinct, and distinct from the ones above. So in this case $\mathcal{L}(\ell_p, \ell_q)$ has at least four non-trivial, proper closed ideals. Later the first named author [8] found two more ideals, again in the range $1 < p < 2 < q < \infty$, namely the ideals generated by operators factoring through the formal inclusion maps $I_{p,2}$ and $I_{2,q}$, respectively. These new ideals lie between $I_{p,q}$ and $\mathcal{FS} \cap \mathcal{J}^{\ell_2}$, and hence the fact they are incomparable shows that $\mathcal{J}^{I_{p,q}} \neq \mathcal{FS} \cap \mathcal{J}^{\ell_2}$, and thus there are at least seven non-trivial, proper, closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ when $1 < p < 2 < q < \infty$.

The main result of this paper is a solution of Pietsch's question in the reflexive range.

Theorem A. *For all $1 < p < q < \infty$ there is a chain of size the continuum consisting of closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ that lie between the ideals $\mathcal{J}^{I_{p,q}}$ and $\mathcal{FS}$.*

Moreover, we obtain the following refinement.

Theorem B. *For all $1 < p < 2 < q < \infty$ there is a chain of size the continuum consisting of closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ that lie between $\mathcal{J}^{I_{p,q}}$ and $\mathcal{J}^{I_{2,q}}$.*

We note that these results provide further examples of separable reflexive Banach spaces X such that $\mathcal{L}(X)$ has continuum many closed ideals. The first such examples were given by A. Pietsch [5].

The paper is organized as follows. In Section 2 we introduce definitions, notations and certain complemented subspaces of ℓ_p that will later lead to new ideals. A crucial rôle is played here by H. P. Rosenthal's famous X_p spaces, which we recall in some detail. We shall use ℓ_p-direct sums of finite-dimensional versions of X_p. The so-called lower fundamental function (defined below) of these direct sums will be computed at the end of Section 2. In Section 3 we prove a key lemma that will be at the heart of the proof of our main results. The latter will be presented in Section 4. We conclude with a list of open problems in Section 5.

Throughout this paper we take the scalar field to be $\mathbb{R}$. All our results can be adapted without difficulty to the complex case.

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2. Definitions and known results

In this section we introduce a large number of definitions as well as some preliminary results. This long section is divided into several subsections to help structure it.

2.1. Operator ideals. Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the space of all bounded, linear operators from X to Y. We write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$, and we let $I_X \in \mathcal{L}(X)$ stand for the identity operator on X. If X and Y have fixed bases (x_i) and (y_i), respectively, such that (y_i) is dominated by (x_i), then we write $I_{X,Y}$ for the formal inclusion $X \to Y$ defined by

$$I_{X,Y}\Big(\sum_i a_i x_i\Big) = \sum_i a_i y_i\,,$$

which is well defined and bounded. When X (respectively, Y) is ℓ_p, $1 \leq p < \infty$, then we write $I_{p,Y}$ (respectively, $I_{X,p}$) instead of $I_{X,Y}$. Similarly, we write $I_{\infty,Y}$ instead of $I_{c_0,Y}$, etc.

By an *ideal of* $\mathcal{L}(X,Y)$ we mean a subspace $\mathcal{J}$ of $\mathcal{L}(X,Y)$ satisfying the ideal property: $ATB \in \mathcal{J}$ for all $A \in \mathcal{L}(Y)$, $T \in \mathcal{J}$ and $B \in \mathcal{L}(X)$. When $X = Y$, this coincides with the usual algebraic notion of an ideal of the algebra $\mathcal{L}(X)$. A closed ideal is an ideal that is closed in the operator norm. Clearly, the closure of an ideal is a closed ideal.

Our notion of ideal is equivalent to Pietsch's notion of *operator ideal* [4]. The latter is a functor that assigns to each pair (V,W) of Banach spaces a subspace $\mathcal{J}(V,W)$ of $\mathcal{L}(V,W)$ such that for all Banach spaces U,V,W,Z and all $A \in \mathcal{L}(W,Z)$, $T \in \mathcal{J}(V,W)$, $B \in \mathcal{L}(U,V)$, we have $ATB \in \mathcal{J}(U,Z)$. This is called a *closed operator ideal* if $\mathcal{J}(V,W)$ is a closed subspace of $\mathcal{L}(V,W)$ for all V,W. Given a (closed) operator ideal $\mathcal{J}$, it is clear that $\mathcal{J}(V,W)$ is then an ideal (respectively, closed ideal), in the above sense, of $\mathcal{L}(V,W)$ for all spaces V and W. Conversely, given a (closed) ideal $\mathcal{J}$ of $\mathcal{L}(X,Y)$ in the above sense, the functor that assigns to (V,W) (the closure of) the set of all finite sums of operators of the form ATB with $A \in \mathcal{L}(Y,W)$, $T \in \mathcal{J}$ and $B \in \mathcal{L}(V,X)$ is a (closed) operator ideal in the sense of Pietsch for which $\mathcal{J}(X,Y) = \mathcal{J}$.

In this paper we shall only deal with closed ideals. Recall that $T \in \mathcal{L}(X,Y)$ is *strictly singular* if it is not an isomorphic embedding on any infinite-dimensional subspace of X, and T is *finitely strictly singular* if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every subspace E of X with $\dim E \geq n$ there exists $x \in E$ such that $\|Tx\| < \varepsilon\|x\|$. We denote by $\mathcal{K}(X,Y) \subset \mathcal{FS}(X,Y) \subset \mathcal{S}(X,Y)$ the ideals of, respectively, compact, finitely strictly singular and strictly singular operators. When $X = Y$, these become $\mathcal{K}(X) \subset \mathcal{FS}(X) \subset \mathcal{S}(X)$. It is not hard to see that these are all closed operator ideals.

For an operator $T\colon U \to V$ between Banach spaces U and V, we let $\mathcal{J}^T = \mathcal{J}^T(X,Y)$ be the closed ideal of $\mathcal{L}(X,Y)$ generated by operators factoring through T. Thus $\mathcal{J}^T$ is the closure in $\mathcal{L}(X,Y)$ of all finite sums of operators of the form ATB, where $A \in \mathcal{L}(V,Y)$ and $B \in \mathcal{L}(X,U)$. When $U = V$ and $T = I_U$, then we write $\mathcal{J}^U$ instead of $\mathcal{J}^{I_U}$.

2.2. ℓ_p spaces. For $1 \leq p \leq \infty$ we denote by p' the conjugate index of p. So we have

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We will denote by $\{e_{p,i} : 1 \leq i < \infty\}$ the unit vector basis of ℓ_p when $1 \leq p < \infty$ and of c_0 when $p = \infty$. For $n \in \mathbb{N}$, the unit vector basis of ℓ_p^n will be denoted by $\{e^{(n)}_{p,j} : 1 \leq j \leq n\}$.

Fix $p \in (1, \infty)$ and define Z_p to be the ℓ_p-direct sum

$$Z_p = \left(\bigoplus_{n=1}^{\infty} \ell_2^n \right)_{\ell_p} .$$

This has a canonical unit vector basis $\{e_{2,j}^{(n)} : n \in \mathbb{N},\ 1 \leq j \leq n\}$. By Khintchine's inequality, the spaces ℓ_2^n, $n \in \mathbb{N}$, are uniformly complemented in ℓ_p. It follows that Z_p is also complemented in ℓ_p, and hence isomorphic to it by Pełczyński's Decomposition Theorem. We fix once and for all an isomorphism $U_p \colon Z_p \to \ell_p$. Although the spaces Z_p and ℓ_p are isomorphic, their canonical unit vector bases $\{e_{2,j}^{(n)} : n \in \mathbb{N},\ 1 \leq j \leq n\}$ and $\{e_{p,i} : 1 \leq i < \infty\}$, respectively, are very different when $p \neq 2$. This is an example of the following convention that we adopt throughout this paper. For us a Banach space means a Banach space together with a fixed basis (which will always be normalized and 1-unconditional). We will use different notation for the same space if we consider more than one basis.

2.3. Current state of affairs. We begin by recalling Volkmann's result [9] that for $1 \leq p < q < \infty$ the algebra $\mathcal{L}(\ell_p \oplus \ell_q)$ has exactly two maximal ideals: the closures of the ideals of operators factoring through ℓ_p and ℓ_q, respectively. Moreover, the set of all non-maximal, proper closed ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$ is in a one-to-one, inclusion-preserving correspondence with the set of all closed ideals of the algebra $\mathcal{L}(\ell_p, \ell_q)$. These results are stated in [4, Theorem 5.3.2]. We also refer the reader to [8, Section 2].

Using the notation established in Section 2.1, the following diagrams summarize what we know about non-trivial closed ideals of $\mathcal{L}(\ell_p, \ell_q)$. When $1 < p < q < \infty$, we have

$$\mathcal{K} \subsetneq \mathcal{J}^{I_{p,q}} \subset \mathcal{FS} \subsetneq \mathcal{L}(\ell_p, \ell_q).$$

Recall that the last two inclusions are due to V. D. Milman [3]. Also, if $\mathcal{J}$ is a proper, closed ideal distinct from $\mathcal{K}$, then $\mathcal{J}$ contains $\mathcal{J}^{I_{p,q}}$.

Combining the main results of [7] and [8], we obtain the following picture in the range $1 < p < 2 < q < \infty$.

$$\begin{tikzcd}
 & & \boxed{\mathcal{J}^{I_{p,2}}} \arrow[dr] & & \boxed{\mathcal{FS}} \arrow[dr] & & \\
\boxed{\mathcal{K}} \arrow[r] & \boxed{\mathcal{J}^{I_{p,q}}} \arrow[ur] \arrow[dr] & \nparallel & \boxed{\mathcal{FS} \cap \mathcal{J}^{\ell_2}} \arrow[ur] \arrow[dr] & \nparallel & \boxed{\mathcal{FS} \vee \mathcal{J}^{\ell_2}} \arrow[r] & \boxed{\mathcal{L}(\ell_p, \ell_q)} \\
 & & \boxed{\mathcal{J}^{I_{2,q}}} \arrow[ur] & & \boxed{\mathcal{J}^{\ell_2}} \arrow[ur] & &
\end{tikzcd}$$

Here arrows stand for inclusion. It is not known whether the ideal $\mathcal{FS} \vee \mathcal{J}^{\ell_2}$, the smallest closed ideal containing $\mathcal{FS}$ and $\mathcal{J}^{\ell_2}$, is proper. All other inclusions are strict. The ideals $\mathcal{K}$ and $\mathcal{J}^{I_{p,q}}$ are the smallest, respectively second smallest non-zero ideals. In [7] it was also shown that any ideal containing an operator not in $\mathcal{FS}$ must contain $\mathcal{J}^{\ell_2}$. It follows that there is no ideal between $\mathcal{FS} \cap \mathcal{J}^{\ell_2}$ and $\mathcal{J}^{\ell_2}$, and that $\mathcal{FS} \vee \mathcal{J}^{\ell_2}$ is the only immediate successor of $\mathcal{FS}$. Two ideals connected by $\nparallel$ are incomparable. It is not known whether $\mathcal{J}^{I_{p,q}}$ is a proper subset of $\mathcal{J}^{I_{p,2}} \cap \mathcal{J}^{I_{2,q}}$.

2.4. Rosenthal's X_p spaces. In 1970 H. P. Rosenthal published an influential paper [6] with important consequences both for the theory of $\mathcal{L}_p$ spaces, and for probability theory. This paper grew out of his study of sequences of independent random variables with mean zero in $L_p = L_p[0,1]$ for $1 < p < \infty$. It led to the introduction of the spaces X_p which we now describe. Let $2 < p < \infty$ and $\boldsymbol{w} = (w_n)_{n=1}^\infty$ be a sequence in $(0,1]$. The space $X_{p,\boldsymbol{w}}$ is the completion of the space c_{00} of finite scalar sequences with respect to the norm

$$\|(a_n)\|_{p,\boldsymbol{w}} = \Big(\sum |a_n|^p\Big)^{\frac{1}{p}} \vee \Big(\sum w_n^2 |a_n|^2\Big)^{\frac{1}{2}}.$$

Rosenthal proved the following [6, Theorem 4]. Let (f_n) be a sequence of independent, symmetric, 3-valued random variables, and let Y_p be the closed linear span of (f_n) in L_p for $1 < p < \infty$. Then Y_p is K_p-complemented in L_p, where K_p is a constant depending only on p. Moreover, if $p > 2$, then Y_p is isomorphic to $X_{p,\boldsymbol{w}}$, where $w_n = \|f_n\|_{L_2}/\|f_n\|_{L_p}$, and $Y_{p'}$ is isomorphic to $X^*_{p,\boldsymbol{w}}$. More precisely, the proof shows that if $1 < p < \infty$ and we assume that $\|f_n\|_{L_p} = 1$ for all $n \in \mathbb{N}$ (as we clearly may), then there is a projection $P_p \colon L_p \to L_p$ onto Y_p given by

$$P_p f = \sum a_n f_n,$$

where

$$a_n = \int_0^1 f(x) f_n(x)\, \mathrm{d}x \cdot \|f_n\|_{L_2}^{-2} \quad \text{for all } n \in \mathbb{N},$$

such that

$$\|(a_n)\|_{p,\boldsymbol{w}} \le \|P_p f\|_{L_p} \le K_p \|(a_n)\|_{p,\boldsymbol{w}} \le K_p \|f\|_{L_p} \quad \text{if } 2 < p < \infty, \tag{2.1}$$

$$\frac{1}{K_p}\|(a_n)\|^*_{p',\boldsymbol{w}} \le \|P_p f\|_{L_p} \le \|(a_n)\|^*_{p',\boldsymbol{w}} \le K_p \|f\|_{L_p} \quad \text{if } 1 < p < 2. \tag{2.2}$$

Here in the case $2 < p < \infty$ we have $w_n = \|f_n\|_{L_2}$ for all $n \in \mathbb{N}$, whereas if $1 < p < 2$, then $w_n = \|f_n\|_{L_2}^{-1}$, and $\|(a_n)\|^*_{p',\boldsymbol{w}}$ denotes the norm of $\sum a_n e_n^*$ in the dual space $X^*_{p',\boldsymbol{w}}$, and (e_n^*) is the sequence biorthogonal to the unit vector basis (e_n) of $X_{p',\boldsymbol{w}}$. Note that P_2 is simply the orthogonal projection of L_2 onto Y_2 and $K_2 = 1$; for $2 < p < \infty$ we obtain P_p as the restriction to L_p of P_2, and for $1 < p < 2$ we have $P_p = P^*_{p'}$ and $K_p = K_{p'}$. It follows from [6, Lemma 2] that in all cases the sequence (f_n) is a 1-unconditional basis of Y_p.

Let $2 < p < \infty$. It is easy to see that $X_{p,\boldsymbol{w}}$ is isomorphic to one of the spaces ℓ_2, ℓ_p and $\ell_2 \oplus \ell_p$ unless (w_n) satisfies

$$\liminf w_n = 0 \quad \text{and} \quad \sum_{n\colon w_n < \varepsilon} w_n^{\frac{2p}{p-2}} = \infty \quad \text{for all } \varepsilon > 0. \tag{2.3}$$

Rosenthal proved that if the sequences (w_n) and (w'_n) both satisfy (2.3), then the corresponding spaces $X_{p,\boldsymbol{w}}$ and $X_{p,\boldsymbol{w}'}$ are isomorphic and distinct from any of the spaces ℓ_2, ℓ_p and $\ell_2 \oplus \ell_p$.

In this paper we shall use finite-dimensional versions of Rosenthal's X_p spaces, and we will only need the result about the existence of well-isomorphic and well-complemented copies in L_p. We begin with some definitions.

Given $2 < p < \infty$, $0 < w \le 1$, $n \in \mathbb{N}$, denote by $E^{(n)}_{p,w}$ the Banach space $(\mathbb{R}^n, \|\cdot\|_{p,w})$, where

$$\big\|(a_j)_{j=1}^n\big\|_{p,w} = \Bigg(\sum_{j=1}^n |a_j|^p\Bigg)^{\frac{1}{p}} \vee w \Bigg(\sum_{j=1}^n |a_j|^2\Bigg)^{\frac{1}{2}}.$$

We write $\{e_j^{(n)} : 1 \leq j \leq n\}$ for the unit vector basis of $E_{p,w}^{(n)}$, and denote by $\{e_j^{(n)*} : 1 \leq j \leq n\}$ the unit vector basis of the dual space $(E_{p,w}^{(n)})^*$, which is biorthogonal to the unit vector basis of $E_{p,w}^{(n)}$.

Given $1 < p < 2$, $0 < w \leq 1$ and $n \in \mathbb{N}$, we fix once and for all a sequence

$$f_j^{(n)} = f_{p,w,j}^{(n)}, \quad 1 \leq j \leq n,$$

of independent symmetric, 3-valued random variables with

$$\|f_j^{(n)}\|_{L_p} = 1 \quad \text{and} \quad \|f_j^{(n)}\|_{L_2} = \frac{1}{w} \quad \text{for } 1 \leq j \leq n.$$

We then define $F_{p,w}^{(n)}$ to be the subspace $\operatorname{span}\{f_j^{(n)} : 1 \leq j \leq n\}$ of L_p. It follows from (2.2) that

$$\frac{1}{K_p}\left\|\sum_{j=1}^n a_j e_j^{(n)*}\right\| \leq \left\|\sum_{j=1}^n a_j f_j^{(n)}\right\|_{L_p} \leq \left\|\sum_{j=1}^n a_j e_j^{(n)*}\right\|, \tag{2.4}$$

where $\{e_j^{(n)*} : 1 \leq j \leq n\}$ is the unit vector basis of the dual space $(E_{p',w}^{(n)})^*$ as defined above. Since the random variables $f_j^{(n)}$ are 3-valued, it follows that $F_{p,w}^{(n)}$ is a subspace of the span of indicator functions of 3^n pairwise disjoint sets. Thus, we can and will think of $F_{p,w}^{(n)}$ as a subspace of $\ell_p^{k_n}$, where $k_n = 3^n$.

Proposition 1. *Let $1 < p < 2$, $0 < w \leq 1$ and $n \in \mathbb{N}$. Then:*

(i) *$\{f_j^{(n)} : 1 \leq j \leq n\}$ is a normalized, 1-unconditional basis of $F_{p,w}^{(n)}$.*

(ii) *There exists a projection $P_{p,w}^{(n)} : \ell_p^{k_n} \to \ell_p^{k_n}$ onto $F_{p,w}^{(n)}$ with $\|P_{p,w}^{(n)}\| \leq K_p$.*

(iii) *For each $1 \leq k \leq n$ and for every $A \subset \{1, \dots, n\}$ with $|A| = k$ we have*

$$\frac{1}{K_p} \cdot \left(k^{\frac{1}{p}} \wedge \frac{1}{w} k^{\frac{1}{2}}\right) \leq \left\|\sum_{j \in A} f_j^{(n)}\right\| \leq k^{\frac{1}{p}} \wedge \frac{1}{w} k^{\frac{1}{2}}.$$

Proof. Statements (i) and (ii) follow from the results of H. P. Rosenthal, [6, Theorem 4] and [6, Lemma 2], that we cited above. By (2.4) we will have proved (iii) if we show that

$$\left\|\sum_{j=1}^k e_j^{(n)*}\right\| = k^{\frac{1}{p}} \wedge \frac{1}{w} k^{\frac{1}{2}},$$

where $\{e_j^{(n)*} : 1 \leq j \leq n\}$ is the unit vector basis of $(E_{p',w}^{(n)})^*$ as defined above. Now, by definition, we have

$$\left\|\sum_{j=1}^k e_j^{(n)*}\right\| = \max\left\{\sum_{j=1}^k a_j : \sum_{j=1}^k |a_j|^{p'} \leq 1 \text{ and } w^2 \sum_{j=1}^k |a_j|^2 \leq 1\right\}.$$

Then by symmetry of $\|\cdot\|_{p',w}$, the maximum occurs when $a_1 = a_2 = \cdots = a_k = t$, say. So

$$\left\|\sum_{j=1}^k e_j^{(n)*}\right\| = \max\{kt : kt^{p'} \leq 1 \text{ and } w^2 k t^2 \leq 1\} = k^{\frac{1}{p}} \wedge \frac{1}{w} k^{\frac{1}{2}},$$

as claimed. □

Remark. We mention two extreme examples. When $w = 1$, then

$$E^{(n)}_{p',w} \cong \ell^n_2,$$

and when $w = n^{-\eta}$ with $\eta = \frac{1}{p} - \frac{1}{2}$, then

$$E^{(n)}_{p',w} \cong \ell^n_{p'}.$$

In both cases the formal identity map is an isometric isomorphism. It follows by (2.4) that if $w = 1$, then

$$F^{(n)}_{p,w} \sim \ell^n_2,$$

and if $w = n^{-\eta}$, then

$$F^{(n)}_{p,w} \sim \ell^n_p.$$

In both cases the formal identity is a K_p-isomorphism.

2.5. The spaces $Y_{p,\boldsymbol{v}}$. Fix $1 < p < 2$, and let $\boldsymbol{v} = (v_n)$ be a decreasing sequence in the interval $(0, 1]$. For each $n \in \mathbb{N}$, let F_n be the subspace $F^{(n)}_{p,v_n}$ of $\ell^{k_n}_p$ with basis $\{f^{(n)}_j : 1 \le j \le n\}$ as defined in Section 2.4 above. We introduce the space $Y_{p,\boldsymbol{v}}$ defined to be the ℓ_p-direct sum

$$Y_{p,\boldsymbol{v}} = \left(\bigoplus_{n=1}^{\infty} F_n\right)_{\ell_p}.$$

This is a K_p-complemented subspace of ℓ_p. Indeed, the diagonal operator

$$P_{p,\boldsymbol{v}} = \operatorname{diag}(P^{(n)}_{p,v_n}) \colon \ell_p \cong \left(\bigoplus_{n=1}^{\infty} \ell^{k_n}_p\right)_{\ell_p} \to \ell_p \cong \left(\bigoplus_{n=1}^{\infty} \ell^{k_n}_p\right)_{\ell_p}$$

is a projection onto $Y_{p,\boldsymbol{v}}$, where $P^{(n)}_{p,v_n}$ is the projection given by Proposition 1 (ii). Furthermore, $Y_{p,\boldsymbol{v}}$ is equipped with the normalized, 1-unconditional basis $\{f^{(n)}_j : n \in \mathbb{N},\ 1 \le j \le n\}$. Note that $Y_{p,\boldsymbol{v}}$, as a complemented subspace of ℓ_p, is isomorphic to ℓ_p. However, we shall never make this identification, and instead consider $Y_{p,\boldsymbol{v}}$ as a complemented subspace of ℓ_p with corresponding projection $P_{p,\boldsymbol{v}}$ fixed as above.

We conclude this section by proving a norm estimate, Lemma 3 below, on sums of basis vectors of $Y_{p,\boldsymbol{v}}$. We begin with fixing some notation. Let X be a Banach space with a fixed (normalized, 1-unconditional) basis (x_i) (finite or infinite). Let $N = \mathbb{N}$ if $\dim(X) = \infty$, and $N = \{1, 2, \dots, \dim(X)\}$ otherwise. We define the *fundamental function* $\varphi_X \colon N \to \mathbb{R}$ *of* X by setting

$$\varphi_X(k) = \sup\left\{\left\|\sum_{i\in A} x_i\right\| : A \subset N,\ |A| \le k\right\}, \quad k \in N.$$

We then extend the definition of φ_X to the real interval $I = \bigcup_{1\le k<\dim(X)} [k, k+1]$ by linear interpolation. The fundamental function plays an important rôle in the study of so-called greedy bases. Here we shall only need the following facts (see, e.g., [1, Section 2]).

Proposition 2. *The functions $t \mapsto \varphi_X(t)$ and $t \mapsto t/\varphi_X(t)$, $t \in I$, are increasing. The concave envelope $\psi \colon I \to \mathbb{R}$ of φ_X, i.e., the (pointwise) smallest concave function dominating φ_X, satisfies $\psi(t) \le 2\varphi_X(t)$ for all $t \in I$.*

We next introduce the *lower fundamental function* $\lambda_X \colon N \to \mathbb{R}$ *of* X defined by

$$\lambda_X(k) = \inf\Big\{\Big\|\sum_{i\in A} x_i\Big\| : A \subset N,\ |A| \geq k\Big\}, \quad k \in N,$$

and extend the definition to I by linear interpolation. It is clear that λ_X is an increasing function on I.

Example. Proposition 1 (iii) shows that

$$\frac{1}{K_p}\cdot\Big(k^{\frac{1}{p}} \wedge \frac{1}{w}k^{\frac{1}{2}}\Big) \leq \lambda_F(k) \leq \varphi_F(k) \leq k^{\frac{1}{p}} \wedge \frac{1}{w}k^{\frac{1}{2}},$$

where $F = F^{(n)}_{p,w}$ and $1 \leq k \leq n$.

We now turn to the estimate on the lower fundamental function of $Y_{p,\boldsymbol{v}}$, as promised.

Lemma 3. *Given* $1 < p < 2$, *let* $\boldsymbol{v} = (v_n)$ *be a decreasing sequence in* $(0,1]$ *such that* $v_n \geq n^{-\eta}$ *for all* $n \in \mathbb{N}$, *where* $\eta = \frac{1}{p} - \frac{1}{2}$. *Then for each* $k \in \mathbb{N}$ *we have*

$$\lambda_{Y_{p,\boldsymbol{v}}}(k) \geq \frac{1}{K_p\cdot\sqrt{2}}\cdot\frac{1}{v_l}\cdot l, \quad \text{where } l = \Big\lfloor\sqrt{\frac{k}{2}}\Big\rfloor.$$

(We put $v_0 = 1$ *to cover the case* $k = 1$.*)*

Proof. Let A be a subset of $\{(n,j) : n \in \mathbb{N},\ 1 \leq j \leq n\}$ with $|A| \geq k$. For each $n \in \mathbb{N}$ set $A_n = A \cap \{(n,j) : 1 \leq j \leq n\}$. By Proposition 1 (iii) we can write $\mathbb{N}$ as the union of disjoint sets L and R, where

$$L = \Big\{n \in \mathbb{N} : \lambda_{F_n}(|A_n|) \geq \frac{1}{K_p}|A_n|^{\frac{1}{p}}\Big\}$$

and

$$R = \Big\{n \in \mathbb{N} : \lambda_{F_n}(|A_n|) \geq \frac{1}{K_p}\frac{1}{v_n}|A_n|^{\frac{1}{2}}\Big\} \setminus L.$$

Then

$$\Big\|\sum_{(n,j)\in A} f_j^{(n)}\Big\|_{Y_{p,\boldsymbol{v}}} \geq \Big(\sum_n \lambda_{F_n}(|A_n|)^p\Big)^{\frac{1}{p}} \geq \frac{1}{K_p}\cdot\Big(\sum_{n\in L}|A_n| + \sum_{n\in R}\Big(\frac{1}{v_n}|A_n|^{\frac{1}{2}}\Big)^p\Big)^{\frac{1}{p}},$$

and hence, using $p < 2$, we obtain

$$K_p\Big\|\sum_{(n,j)\in A} f_j^{(n)}\Big\|_{Y_{p,\boldsymbol{v}}} \geq \Big(\sum_{n\in L}|A_n| + \Big(\sum_{n\in R}\frac{1}{v_n^2}|A_n|\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}. \tag{2.5}$$

Set $l = \lfloor\sqrt{k/2}\rfloor$. Since $\sum_n |A_n| = |A| \geq k$, either $\sum_{n\in L}|A_n| \geq l^2$ or $\sum_{n\in R}|A_n| \geq l^2$. In the former case, inequality (2.5) immediately gives

$$K_p\Big\|\sum_{(n,j)\in A} f_j^{(n)}\Big\|_{Y_{p,\boldsymbol{v}}} \geq l^{\frac{2}{p}} \geq \frac{1}{v_l}\cdot l, \tag{2.6}$$

where the second inequality follows from the assumption that $v_n \geq n^{-\eta}$ for all $n \in \mathbb{N}$.

We now consider the case when $\sum_{n\in R}|A_n| \geq l^2$. Choose $s \in \mathbb{N}$ and $0 \leq s' \leq s$ such that

$$\sum_{n\in R}|A_n| = \sum_{n=1}^{s-1} n + s'.$$

For $n \in \mathbb{N}$ and $1 \leq j \leq n$ set $v_{n,j} = v_n$. Since (v_n) is decreasing, summing $1/v_{n,j}^2$ over the set $\bigcup_{n\in R} A_n$ is minimized when $\bigcup_{n\in R} A_n$ is an initial segment of $\{(n,j) : n \in \mathbb{N},\ 1 \leq j \leq n\}$ in the lexicographic order. Thus we have

$$\sum_{n\in R} \frac{1}{v_n^2}|A_n| \geq \sum_{n=1}^{s-1} \frac{1}{v_n^2} n + \frac{1}{v_s^2} s' \geq \frac{1}{v_l^2}\left(\sum_{n=l}^{s-1} n + s'\right) \geq \frac{1}{v_l^2} \cdot \frac{l^2}{2},$$

where we used $\sum_{n=1}^{l-1} n \leq \frac{l^2}{2}$ in the last inequality. Hence by (2.5) we obtain

$$K_p \Big\| \sum_{(n,j)\in A} f_j^{(n)} \Big\|_{Y_{p,v}} \geq \frac{1}{\sqrt{2}} \cdot \frac{1}{v_l} \cdot l. \tag{2.7}$$

The claim now follows from (2.6) and (2.7) above. □

3. The key lemma

This section is entirely devoted to a result that will play a central rôle in distinguishing closed ideals. It roughly says that if one has a bounded operator and the fundamental function of the domain is asymptotically smaller than the lower fundamental function of the range space, then a large proportion of basis vectors must map to 'flat' vectors.

Lemma 4. *Let Y be an infinite-dimensional Banach space with a normalized,* 1*-unconditional basis (f_j). For each integer $m \in \mathbb{N}$ let G_m be an m-dimensional Banach space with a normalized,* 1*-unconditional basis $\{g_i^{(m)} : 1 \leq i \leq m\}$. Assume that*

$$\lim_{k\to\infty} \sup_{m\geq k} \frac{\varphi_{G_m}(k)}{k} = 0 \tag{3.1}$$

and

$$\lim_{m\to\infty} \frac{\varphi_{G_m}(m)}{\lambda_Y(cm)} = 0 \quad \text{for all } c > 0. \tag{3.2}$$

If $(B_m\colon G_m \to Y)_{m=1}^{\infty}$ is a sequence of operators with $\sup_m \|B_m\| \leq 1$, then

$$\frac{1}{m}\sum_{i=1}^{m} \big\| B_m(g_i^{(m)}) \big\|_\infty \to 0 \quad \text{as } m \to \infty.$$

Here $\|y\|_\infty = \sup_j |y_j|$ for $y = \sum_j y_j f_j \in Y$.

Remark. Before proving our lemma let us look at the extreme case. Assume that for each $m \in \mathbb{N}$ we are given a linear operator B_m from ℓ_∞^m to ℓ_1 with $\|B_m\| \leq 1$. In that special case we can easily deduce our claim from Grothendieck's inequality. Indeed, fixing $m \in \mathbb{N}$, we can write B_m as a matrix $B_m = (B_m(j,i))$, $i = 1,\dots,m$, $j \in \mathbb{N}$, with

$$\sup\left\{\sum_{i=1}^{m}\sum_{j=1}^{\infty} t_j B_m(j,i)s_i : |s_i|, |t_j| \le 1 \text{ for } 1 \le i \le m,\ 1 \le j < \infty\right\} = \|B_m\| \le 1,$$

and we then have to show that

$$\frac{1}{m}\sum_{i=1}^{m}\|B_m(e_{\infty,i}^{(m)})\|_\infty = \frac{1}{m}\sum_{i=1}^{m}\max_{j\in\mathbb{N}}|B_m(j,i)| \to 0 \quad \text{as } m \to \infty.$$

Now Grothendieck's inequality implies that

$$\sum_{i=1}^{m}\sum_{j=1}^{\infty} B_m(j,i)\langle y_j, x_i\rangle \le K_G,$$

whenever x_i, $i = 1,\dots,m$, and y_j, $j \in \mathbb{N}$, are elements of the unit ball of a Hilbert space H, and where K_G denotes the Grothendieck constant. We choose for each $i = 1,\dots,m$ an integer $j_i \in \mathbb{N}$ such that

$$|B_m(j_i,i)| = \max_{j\in\mathbb{N}}|B_m(j,i)|.$$

We then let $H = \ell_2^m$ and $x_i = e_{2,i}^{(m)}$ for $i = 1,\dots,m$. For each $j \in \mathbb{N}$ we define a vector

$$\tilde{y}_j = \sum_{i=1}^{m}\tilde{y}_j(i)e_{2,i}^{(m)}$$

in ℓ_2^m as follows:

$$\tilde{y}_j(i) = \begin{cases} \operatorname{sign}(B_m(j_i,i)) & \text{if } j = j_i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\|\tilde{y}_j\|_{\ell_2^m} \le \sqrt{m}$, and so $y_j = \tilde{y}_j/\sqrt{m}$ is in the unit ball of ℓ_2^m for each $j \in \mathbb{N}$. It follows that

$$\begin{aligned} K_G &\ge \sum_{i=1}^{m}\sum_{j=1}^{\infty} B_m(j,i)\langle y_j, x_i\rangle \\ &= \sum_{i=1}^{m}\sum_{j=1}^{\infty} B_m(j,i)y_j(i) \\ &= \frac{1}{\sqrt{m}}\sum_{i=1}^{m}\big|B_m(j_i,i)\big| \\ &= \frac{1}{\sqrt{m}}\sum_{i=1}^{m}\max_{j\in\mathbb{N}}\big|B_m(j,i)\big|, \end{aligned}$$

which yields our claim in this special case.

Proof of Lemma 4. Fix $\varrho > 0$. By (3.1) there exists an integer $k_0 \in \mathbb{N}$ such that

$$\frac{\varphi_{G_m}(k)}{k} < \frac{\varrho}{2} \quad \text{for all } m \ge k \ge k_0.$$

Set $c = \varrho k_0^{-1}$. By (3.2) we may choose $m_0 \in \mathbb{N}$ such that

$$m_0 > k_0 \quad \text{and} \quad \frac{\varphi_{G_m}(m)}{\lambda_Y(cm)} < \varrho \quad \text{for all } m > m_0.$$

Now fix $m > m_0$ and set

$$A = \big\{i \in \{1, 2, \dots, m\} : \big\| B_m\big(g_i^{(m)}\big)\big\|_\infty > \varrho\big\}.$$

We will show that $|A| \leq \varrho m$. It will then follow that

$$\frac{1}{m}\sum_{i=1}^{m} \big\| B_m\big(g_i^{(m)}\big)\big\|_\infty \leq \varrho + \frac{|A|}{m} \leq 2\varrho,$$

and since $m > m_0$ and $\varrho > 0$ were arbitrary, the proof of the lemma will be complete. To show that $|A| \leq \varrho m$ we argue by contradiction, and assume that $|A| > \varrho m$. For each $i \in A$ fix $j_i \in \mathbb{N}$ such that

$$\big|\big[B_m\big(g_i^{(m)}\big)\big]_{j_i}\big| \geq \varrho.$$

Here $[y]_j$ denotes, for $y \in Y$, the j-th coordinate of y with respect to the basis $(f_j)_{j=1}^{\infty}$. We then set $\tilde{A} = \{j_i : i \in A\}$, and for $j \in \tilde{A}$ we let $A_j = \{i \in A : j_i = j\}$. We shall now obtain a number of inequalities that will eventually lead to a contradiction.

Fix $j \in \tilde{A}$ and for each $i = 1, \dots, m$ let ε_i be the sign of $[B_m(g_i^{(m)})]_j$. Since $\|B_m\| \leq 1$ and $(g_i^{(m)})$ is 1-unconditional, we have

$$\begin{aligned}
\varphi_{G_m}(|A_j|) &\geq \Big\| \sum_{i \in A_j} \varepsilon_i g_i^{(m)} \Big\|_{G_m} \\
&\geq \Big\| \sum_{i \in A_j} \varepsilon_i B_m\big(g_i^{(m)}\big) \Big\|_{Y} \\
&\geq \Big[\sum_{i \in A_j} \varepsilon_i B_m\big(g_i^{(m)}\big) \Big]_{j} \\
&\geq |A_j|\varrho.
\end{aligned}$$

Let ψ be the concave envelope of φ_{G_m}. Since A is the disjoint union of the sets A_j, $j \in \tilde{A}$, we obtain

$$\begin{aligned}
|A| &= \sum_{j \in \tilde{A}} |A_j| \\
&\leq \varrho^{-1} \sum_{j \in \tilde{A}} \varphi_{G_m}(|A_j|) \\
&\leq \varrho^{-1} \sum_{j \in \tilde{A}} \psi(|A_j|) \\
&\leq \varrho^{-1} |\tilde{A}| \cdot \psi\Big(\frac{|A|}{|\tilde{A}|}\Big) \qquad \text{(by the concavity of } \psi) \\
&\leq 2\varrho^{-1} |\tilde{A}| \cdot \varphi_{G_m}\Big(\frac{|A|}{|\tilde{A}|}\Big) \qquad \text{(by Proposition 2).}
\end{aligned}$$

Now if $|A| > k_0|\tilde{A}|$, then it follows from the above that $\frac{\varphi_{G_m}(k_0)}{k_0} \geq \frac{\varrho}{2}$ which contradicts the choice of k_0. Thus

$$|A| \leq k_0|\tilde{A}|. \tag{3.3}$$

We next fix independent Rademacher random variables $(r_i)_{i\in A}$, and show the estimate

$$\mathbb{E}\Big|\sum_{i\in A} r_i\big[B_m\big(g_i^{(m)}\big)\big]_j\Big| \geq \varrho \quad \text{for all } j \in \tilde{A}. \tag{3.4}$$

Note that the expectation here is simply the average over all choices of $\pm$ signs (and the use of Jensen's inequality below is nothing else but the triangle-inequality). However, the use of Rademachers to express averages is a common device in Banach space theory, and does make the calculation somewhat more transparent.

To see (3.4) fix $j \in \tilde{A}$ and set $y_i = [B_m(g_i^{(m)})]_j$ for $i \in A$. By the definition of $\tilde{A}$ there is an $i_0 \in A$ such that $j_{i_0} = j$, and hence $|y_{i_0}| \geq \varrho$. Thus

$$\begin{aligned}
\mathbb{E}\Big|\sum_{i\in A} r_i y_i\Big| &= \mathbb{E}\Big|\sum_{i\in A} r_{i_0} r_i y_i\Big| \\
&= \mathbb{E}\Big|y_{i_0} + \sum_{i\in A, i\neq i_0} r_{i_0} r_i y_i\Big| \\
&\geq \Big|y_{i_0} + \sum_{i\in A, i\neq i_0} \mathbb{E}(r_{i_0} r_i) y_i\Big| \\
&= |y_{i_0}| \geq \varrho,
\end{aligned}$$

using Jensen's inequality in the third line. We next obtain

$$\begin{aligned}
\varphi_{G_m}(|A|) &\geq \mathbb{E}\Big\|\sum_{i\in A} r_i B_m\big(g_i^{(m)}\big)\Big\|_Y && (\text{as } \|B_m\| \leq 1) \\
&= \mathbb{E}\Big\|\sum_j \Big|\sum_{i\in A} r_i\big[B_m\big(g_i^{(m)}\big)\big]_j\Big| f_j\Big\|_Y && (\text{as } (f_j) \text{ is 1-unconditional}) \\
&\geq \Big\|\sum_j \mathbb{E}\Big|\sum_{i\in A} r_i\big[B_m\big(g_i^{(m)}\big)\big]_j\Big| f_j\Big\|_Y && (\text{by Jensen's inequality}) \\
&\geq \varrho\Big\|\sum_{j\in\tilde{A}} f_j\Big\|_Y && (\text{using (3.4) and the 1-unconditionality of } (f_j)) \\
&\geq \varrho\lambda_Y(|\tilde{A}|).
\end{aligned}$$

Recall that $c = \varrho k_0^{-1}$ and $A \subset \{1,\ldots,m\}$ with $|A| > \varrho m$. So $|A| \leq m$, and by (3.3), $|\tilde{A}| \geq cm$. Thus, the above gives

$$\varrho \leq \frac{\varphi_{G_m}(|A|)}{\lambda_Y(|\tilde{A}|)} \leq \frac{\varphi_{G_m}(m)}{\lambda_Y(cm)} < \varrho$$

by the choice of m_0. This contradiction completes the proof. □

4. Proof of the main results

In the previous section we described a situation when images of basis vectors are on average 'flat'. Here we begin with a calculation (Lemma 5 below) that shows that certain formal inclusion maps reduce the norm of 'flat' vectors. We then introduce, in the special case $1 < p < 2$ and $p < q < \infty$, a class of closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ parametrised by decreasing sequences in $(0, 1]$. Theorem 6 and Corollary 7 show when these ideals are distinct from one another. This will lead to a proof of our main result, Theorem A. In the rest of the section we follow a similar strategy and establish Theorem B.

Lemma 5. *Given $1 < p < 2$ and $p < q < \infty$, let $n \in \mathbb{N}$, $v \in (0, 1]$, and $F = F^{(n)}_{p,v}$ with basis $\{f^{(n)}_j : 1 \le j \le n\}$. Let*

$$y = \sum_{j=1}^{n} y_j f^{(n)}_j \in F$$

with $\|y\|_F \le 1$, and let

$$\tilde{y} = \sum_{j=1}^{n} y_j e^{(n)}_{2,j} \in \ell^n_2.$$

If $\|y\|_\infty = \max_j |y_j| \le \sigma \le 1$ and $v \le \sigma^{\frac{1}{2}-\frac{1}{p'}}$, then

$$\|\tilde{y}\|^q_{\ell^n_2} \le \max\{C^p_p, K^q_p\} \cdot \sigma^r \cdot \|y\|^p_F,$$

where C_p is the cotype-2 constant of ℓ_p and $r = \min\{\frac{q}{2} - \frac{p}{2}, \frac{q}{2} - \frac{q}{p'}\}$.

Here we recall that for $1 \le p \le 2$ the Banach space $L_p[0, 1]$, and hence ℓ_p, has cotype 2. This is a consequence of Khintchine's inequality. See for example [2, Definition 1.e.12].

Proof. If $\|\tilde{y}\|_{\ell^n_2} \le \sqrt{\sigma}$, then

$$\|\tilde{y}\|^q_{\ell^n_2} = \|\tilde{y}\|^{q-p}_{\ell^n_2} \cdot \|\tilde{y}\|^p_{\ell^n_2} \le \sigma^{\frac{q}{2}-\frac{p}{2}} \cdot \|\tilde{y}\|^p_{\ell^n_2} \le C^p_p \cdot \sigma^{\frac{q}{2}-\frac{p}{2}} \cdot \|y\|^p_F,$$

where we use the fact, an easy consequence of the definition of cotype, that in ℓ_p a normalized, 1-unconditional basis C_p-dominates the unit vector basis of ℓ_2. So the claim holds in this case.

Now assume that $\|\tilde{y}\|_{\ell^n_2} > \sqrt{\sigma}$. Set $z_j = \frac{y_j}{\|\tilde{y}\|_{\ell^n_2}}$ for $1 \le j \le n$, and let

$$\tilde{z} = \sum_{j=1}^{n} z_j e^{(n)}_{2,j}.$$

Then $\|\tilde{z}\|_{\ell^n_2} = 1$ and $\langle \tilde{y}, \tilde{z} \rangle = \|\tilde{y}\|_{\ell^n_2}$. Note also that $|z_j| \le \sqrt{\sigma}$ for all j. So we have

$$\left(\sum_{j=1}^{n} |z_j|^{p'}\right)^{\frac{1}{p'}} = \left(\sum_{j=1}^{n} |z_j|^{p'-2} \cdot |z_j|^2\right)^{\frac{1}{p'}} \le \sigma^{\frac{1}{2}-\frac{1}{p'}} \cdot \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{p'}} = \sigma^{\frac{1}{2}-\frac{1}{p'}}.$$

On the other hand, we have

$$v \cdot \|\tilde{z}\|_{\ell^n_2} = v \le \sigma^{\frac{1}{2}-\frac{1}{p'}}$$

by assumption. It follows that

$$\|z\|_{p',v} \le \sigma^{\frac{1}{2}-\frac{1}{p'}},$$

where
$$z = \sum_{j=1}^{n} z_j e_j^{(n)},$$
and $\{e_j^{(n)} : 1 \le j \le n\}$ is the unit vector basis of $E = E_{p',v}^{(n)} = (\mathbb{R}^n, \|\cdot\|_{p',v})$. Hence by (2.4) we have
$$\|\tilde{y}\|_{\ell_2^n} = \langle \tilde{y}, \tilde{z}\rangle \le \|(y_j)\|_{p',v}^* \cdot \|(z_j)\|_{p',v} \le K_p \cdot \sigma^{\frac{1}{2}-\frac{1}{p'}} \cdot \|y\|_F.$$
It follows that
$$\|\tilde{y}\|_{\ell_2^n}^q \le K_p^q \cdot \sigma^{\frac{q}{2}-\frac{q}{p'}} \cdot \|y\|_F^q,$$
which proves the claim since $\|y\|_F \le 1$, and so $\|y\|_F^q \le \|y\|_F^p$. □

Fix $1 < p < 2$ and $p < q < \infty$. Let $\boldsymbol{v} = (v_n)$ be a decreasing sequence in $(0,1]$. In Section 2.5 we introduced the complemented subspace $Y_{\boldsymbol{v}} = Y_{p,\boldsymbol{v}}$ of ℓ_p with corresponding projection $P_{\boldsymbol{v}} = P_{p,\boldsymbol{v}}$. As already mentioned in the proof above, for each $n \in \mathbb{N}$, the unit vector basis of ℓ_2^n is C_p-dominated by the normalized, 1-unconditional basis $\{f_j^{(n)} : 1 \le j \le n\}$ of the subspace $F_n = F_{p,v_n}^{(n)}$. Thus the formal inclusion map
$$I_{Y_{\boldsymbol{v}},Z_q} \colon Y_{\boldsymbol{v}} = \left(\bigoplus_{n=1}^{\infty} F_n\right)_{\ell_p} \to Z_q = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_q}$$
given by
$$I_{Y_{\boldsymbol{v}},Z_q}\big(f_j^{(n)}\big) = e_{2,j}^{(n)}$$
is well defined and bounded. This defines the closed ideal $\mathcal{J}^{I_{Y_{\boldsymbol{v}},Z_q}}$ of $\mathcal{L}(\ell_p, \ell_q)$ generated by operators factoring through $I_{Y_{\boldsymbol{v}},Z_q}$. In Section 2.2 we also fixed an isomorphism $U_q \colon Z_q \to \ell_q$. Note that the operator $T_{\boldsymbol{v}} = U_q \circ I_{Y_{\boldsymbol{v}},Z_q} \circ P_{\boldsymbol{v}}$ belongs to the ideal $\mathcal{J}^{I_{Y_{\boldsymbol{v}},Z_q}}$. The next result establishes conditions on two sequences $\boldsymbol{v}$ and $\boldsymbol{w}$ which imply that $T_{\boldsymbol{w}} \notin \mathcal{J}^{I_{Y_{\boldsymbol{v}},Z_q}}$.

Theorem 6. *Fix $1 < p < 2$ and $p < q < \infty$. Let $\boldsymbol{v} = (v_n)$ and $\boldsymbol{w} = (w_n)$ be decreasing sequences in $(0,1]$. Consider $Y_{\boldsymbol{v}}$, $I_{Y_{\boldsymbol{v}},Z_q}$, $T_{\boldsymbol{w}}$ as above. Assume that $v_n \ge n^{-\eta}$ and $w_n \ge n^{-\eta}$ for all $n \in \mathbb{N}$, where $\eta = \frac{1}{p} - \frac{1}{2}$. Further assume that*
$$\frac{v_{\sqrt{cn}}}{w_n} \to 0 \quad \text{as } n \to \infty \quad \text{for all } c > 0 \tag{4.1}$$
(where we simplify notation by letting $v_0 = 0$ and $v_x = v_{\lfloor x \rfloor}$ for a positive real number x). Then $T_{\boldsymbol{w}} \notin \mathcal{J}^{I_{Y_{\boldsymbol{v}},Z_q}}$.

Proof. For each $n \in \mathbb{N}$ let
$$F_n = F_{p,v_n}^{(n)} \quad \text{and} \quad G_n = F_{p,w_n}^{(n)}$$
with unit vector bases $\{f_j^{(n)} : 1 \le j \le n\}$ and $\{g_j^{(n)} : 1 \le j \le n\}$, respectively. Thus
$$f_j^{(n)} = f_{p,v_n,j}^{(n)} \quad \text{and} \quad g_j^{(n)} = f_{p,w_n,j}^{(n)} \quad \text{for } 1 \le j \le n$$
using the notation introduced in Section 2.4. To simplify notation we write $Y = Y_{\boldsymbol{v}}$, $Z = Z_q$, $U = U_q$ and $T = T_{\boldsymbol{w}}$. Thus $U \colon Z \to \ell_q$ is an isomorphism,
$$I_{Y,Z} \colon Y = \left(\bigoplus_{n=1}^{\infty} F_n\right)_{\ell_p} \to Z = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_q}$$

is given by

$$I_{Y,Z}\big(f_j^{(n)}\big) = e_{2,j}^{(n)},$$

and

$$T : \ell_p = \left(\bigoplus_{m=1}^{\infty} \ell_p^{k_m} \right)_{\ell_p} \to \ell_q$$

is the composite

$$T = U \circ I_{Y_{\boldsymbol{w}},Z} \circ P_{\boldsymbol{w}}.$$

Note that $T(g_i^{(m)}) = U(e_{2,i}^{(m)})$ for each $m \in \mathbb{N}$ and $i = 1, \dots, m$.

We need to show that $T \notin \mathcal{J}^{I_{Y,Z}}$. We achieve this by finding a separating functional $\Phi \in \mathcal{L}(\ell_p, \ell_q)^*$ as follows. For each $m \in \mathbb{N}$ we define $\Phi_m \in \mathcal{L}(\ell_p, \ell_q)^*$ by setting

$$\Phi_m(V) = \frac{1}{m} \sum_{i=1}^{m} \big\langle e_{2,i}^{(m)}, U^{-1} V(g_i^{(m)}) \big\rangle, \quad V \in \mathcal{L}(\ell_p, \ell_q).$$

As $\|\Phi_m\| \le \|U^{-1}\|$ for all m, the sequence (Φ_m) has a ω^*-accumulation point Φ in $\mathcal{L}(\ell_p, \ell_q)^*$. Note that $\Phi_m(T) = 1$ for all m, and hence $\Phi(T) = 1$. The proof will be complete if we can show that $\mathcal{J}^{I_{Y,Z}}$ is contained in the kernel of Φ. To see this, fix $A \in \mathcal{L}(Z, \ell_q)$ and $B \in \mathcal{L}(\ell_p, Y)$ with $\|A\| \le 1$ and $\|B\| \le 1$. It is sufficient to show that

$$\Phi_m(A I_{Y,Z} B) \to 0 \quad \text{as } m \to \infty. \tag{4.2}$$

Let $B_m : G_m \to Y$ denote the restriction to G_m of B. We shall use Lemma 4 to show that

$$\frac{1}{m} \sum_{i=1}^{m} \big\| B_m\big(g_i^{(m)}\big) \big\|_\infty \to 0 \quad \text{as } m \to \infty. \tag{4.3}$$

Recall that for $y = \sum_{n=1}^{\infty} \sum_{j=1}^{n} y_{n,j} f_j^{(n)}$ in Y we let

$$\|y\|_\infty = \sup_{n \in \mathbb{N},\, 1 \le j \le n} |y_{n,j}|.$$

By Proposition 1 (iii) we have

$$\varphi_{G_m}(k) \le k^{\frac{1}{p}} \quad \text{for all } 1 \le k \le m,$$

and so condition (3.1) of Lemma 4 certainly holds. Now by Lemma 3 we have

$$\lambda_Y(k) \ge \frac{1}{3K_p} \cdot \frac{1}{v_{\sqrt{k/2}}} \cdot \sqrt{k}$$

for all large k. On the other hand, by Proposition 1 (iii) we have

$$\varphi_{G_m}(m) \le \frac{1}{w_m} m^{\frac{1}{2}} \quad \text{for all } m.$$

So for any $c > 0$ and for all sufficiently large $m \in \mathbb{N}$, it follows that

$$\frac{\varphi_{G_m}(m)}{\lambda_Y(cm)} \le C \cdot \frac{v_{\sqrt{c'm}}}{w_m},$$

where C and c' are constants depending only on c (and p). Thus, by assumption (4.1), condition (3.2) also holds, and Lemma 4 applies. This completes the proof of (4.3). To see (4.2), fix $\varrho \in (0,1)$ and choose $n_0 \in \mathbb{N}$ such that $v_n \leq \varrho^{\frac{1}{2}-\frac{1}{p'}}$ for all $n \geq n_0$. This is possible, since by (4.1) we have $v_n \to 0$ as $n \to \infty$. Note that

$$|\Phi_m(AI_{Y,Z}B)| = \frac{1}{m}\left|\sum_{i=1}^m \big\langle A^*\big(U^{-1}\big)^* e_{2,i}^{(m)}, I_{Y,Z}B_m\big(g_i^{(m)}\big)\big\rangle\right| \leq \frac{1}{m}\sum_{i=1}^m \big\|U^{-1}\big\| \cdot \big\|I_{Y,Z}B_m\big(g_i^{(m)}\big)\big\|_Z. \tag{4.4}$$

For each $m, i \in \mathbb{N}$ with $1 \leq i \leq m$ we put

$$\sigma_i^{(m)} = \varrho \vee \big\|B_m\big(g_i^{(m)}\big)\big\|_\infty.$$

We now fix $1 \leq i \leq m$ and, to simplify notation, we temporarily write $\sigma = \sigma_i^{(m)}$ and

$$x = B_m\big(g_i^{(m)}\big) = \sum_{n=1}^\infty \sum_{j=1}^n x_{n,j} f_j^{(n)}.$$

Note that $v_n \leq \sigma^{\frac{1}{2}-\frac{1}{p'}}$ for all $n \geq n_0$. Hence by Lemma 5 we have

$$\left(\sum_{j=1}^n |x_{n,j}|^2\right)^{\frac{q}{2}} \leq M \cdot \sigma^r \cdot \left\|\sum_{j=1}^n x_{n,j} f_j^{(n)}\right\|_{F_n}^p \quad \text{for all } n \geq n_0,$$

where $M = \max\{C_p^p, K_p^q\}$ and $r = \min\{\frac{q}{2} - \frac{p}{2}, \frac{q}{2} - \frac{q}{p'}\}$. It follows that

$$\begin{aligned}
\big\|I_{Y,Z}B_m\big(g_i^{(m)}\big)\big\|_Z &= \|I_{Y,Z}(x)\|_Z \\
&= \left(\sum_{n=1}^\infty \left(\sum_{j=1}^n |x_{n,j}|^2\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{n=1}^{n_0} \left(\sum_{j=1}^n |x_{n,j}|^2\right)^{\frac{q}{2}}\right)^{\frac{1}{q}} + M^{\frac{1}{q}} \cdot \sigma^{\frac{r}{q}} \cdot \left(\sum_{n>n_0} \left\|\sum_{j=1}^n x_{n,j} f_j^{(n)}\right\|_{F_n}^p\right)^{\frac{1}{q}} \\
&\leq \|x\|_\infty \cdot \left(\sum_{n=1}^{n_0} n^{\frac{q}{2}}\right)^{\frac{1}{q}} + M^{\frac{1}{q}} \cdot \sigma^{\frac{r}{q}} \cdot \|x\|_Y^{\frac{p}{q}} \\
&\leq \|x\|_\infty \cdot N + M^{\frac{1}{q}} \cdot \sigma^{\frac{r}{q}} \\
&= N \cdot \big\|B_m\big(g_i^{(m)}\big)\big\|_\infty + M^{\frac{1}{q}} \cdot \big(\sigma_i^{(m)}\big)^{\frac{r}{q}},
\end{aligned}$$

where we put $N = \big(\sum_{n=1}^{n_0} n^{\frac{q}{2}}\big)^{\frac{1}{q}}$. Hence, using (4.4), we obtain

$$\begin{aligned}
\big\|U^{-1}\big\|^{-1} \cdot |\Phi_m(AI_{Y,Z}B)| &\leq N \cdot \frac{1}{m}\sum_{i=1}^m \big\|B_m\big(g_i^{(m)}\big)\big\|_\infty + M^{\frac{1}{q}} \cdot \frac{1}{m}\sum_{i=1}^m \big(\sigma_i^{(m)}\big)^{\frac{r}{q}} \\
&\leq N \cdot \frac{1}{m}\sum_{i=1}^m \big\|B_m\big(g_i^{(m)}\big)\big\|_\infty + M^{\frac{1}{q}} \cdot \left(\frac{1}{m}\sum_{i=1}^m \sigma_i^{(m)}\right)^{\frac{r}{q}}.
\end{aligned}$$

To see the second inequality note that $\frac{r}{q} < \frac{1}{2}$, and so the function $t \mapsto t^{\frac{r}{q}}$ is concave. Now it follows from (4.3) that

$$\limsup_m |\Phi_m(AI_{Y,Z}B)| \leq \left\|U^{-1}\right\| \cdot M^{\frac{1}{q}} \cdot \varrho^{\frac{r}{q}}.$$

Since $\varrho > 0$ was arbitrary, the proof of (4.2), and hence of the theorem, is complete. □

Corollary 7. *Let $1 < p < 2$ and $p < q < \infty$. Let $\boldsymbol{v} = (v_n)$ and $\boldsymbol{w} = (w_n)$ be decreasing sequences in $(0,1]$ bounded below by $n^{-\eta}$, $\eta = \frac{1}{p} - \frac{1}{2}$, and satisfying condition (4.1). Then $\mathcal{J}^{I_{Y_{\boldsymbol{v}},Z_q}} \subsetneq \mathcal{J}^{I_{Y_{\boldsymbol{w}},Z_q}}$.*

Proof. It follows from (4.1) that eventually $v_n \leq w_n$. Hence, using the notation of the proof of Theorem 6, the basis $(f_j^{(n)})$ of F_n K_p-dominates the basis $(g_j^{(n)})$ of G_n for all large n. Indeed, this follows from (2.4). Thus, $I_{Y_{\boldsymbol{v}},Z_q}$ factors through $I_{Y_{\boldsymbol{w}},Z_q}$ via the formal inclusion map $I_{Y_{\boldsymbol{v}},Y_{\boldsymbol{w}}}$, and thus $\mathcal{J}^{I_{Y_{\boldsymbol{v}},Z_q}} \subset \mathcal{J}^{I_{Y_{\boldsymbol{w}},Z_q}}$. The claim now follows immediately from Theorem 6 since $T_{\boldsymbol{w}} \in \mathcal{J}^{I_{Y_{\boldsymbol{w}},Z_q}}$. □

Before proving Theorem A we need to show that certain maps are finitely strictly singular.

Proposition 8. *Fix $1 < p < 2$ and $p < q < \infty$. Let $\boldsymbol{v} = (v_n)$ be a decreasing sequence in $(0,1]$ bounded below by $n^{-\eta}$, $\eta = \frac{1}{p} - \frac{1}{2}$, such that $v_n \to 0$ as $n \to \infty$. Set $Y = Y_{p,\boldsymbol{v}}$. Then the formal inclusion maps I_{Y,Z_q} and $I_{Z_{q'},Y^*}$ are finitely strictly singular.*

Proof. For each $n \in \mathbb{N}$ let

$$F_n = F_{p,v_n}^{(n)} \quad \text{and} \quad E_n = (\mathbb{R}^n, \|\cdot\|_{p',v_n})$$

with unit vector bases $\{f_j^{(n)} : 1 \leq j \leq n\}$ and $\{e_j^{(n)} : 1 \leq j \leq n\}$, respectively. We first prove that I_{Y,Z_q} is finitely strictly singular. Fix $\varepsilon > 0$. Choose $\varrho \in (0,1)$ such that

$$\varrho + M^{\frac{1}{q}} \cdot \varrho^{\frac{r}{q}} < \varepsilon,$$

where, as before, $M = \max\{C_p^p, K_p^q\}$, $r = \min\{\frac{q}{2} - \frac{p}{2}, \frac{q}{2} - \frac{q}{p'}\}$ and C_p is the cotype-2 constant of ℓ_p. Next fix $n_0 \in \mathbb{N}$ such that

$$v_n \leq \varrho^{\frac{1}{2} - \frac{1}{p'}} \quad \text{for all } n \geq n_0.$$

Set $N = (\sum_{n=1}^{n_0} n^{\frac{q}{2}})^{\frac{1}{q}}$. Finally, choose $d \in \mathbb{N}$ such that

$$K_p \cdot \frac{2v_d}{d} \cdot N < \varrho.$$

Now let H be a subspace of Y of dimension at least $2d^2$. By a result of V. D. Milman [3] (see also [7, Lemma 3.4]), there exists a non-zero vector

$$x = \sum_{n=1}^{\infty} \sum_{j=1}^{n} x_{n,j} f_j^{(n)} \in H$$

such that

$$|x_{m,i}| = \|x\|_\infty = \sup_{n \in \mathbb{N},\, 1 \leq j \leq n} |x_{n,j}|$$

for at least $2d^2$ pairs (m,i). Hence by Lemma 3, assuming as we may that $\|x\|_Y = 1$, we have

$$1 = \|x\|_Y \geq \|x\|_\infty \cdot \lambda_Y\big(2d^2\big) \geq \|x\|_\infty \cdot \frac{1}{K_p\sqrt{2}} \cdot \frac{d}{v_d}. \tag{4.5}$$

Set $\sigma = \varrho \vee \|x\|_\infty$. As in the proof of Theorem 6, we obtain

$$\|I_{Y,Z}(x)\| \leq N \cdot \|x\|_\infty + M^{\frac{1}{q}} \cdot \sigma^{\frac{r}{q}}.$$

By (4.5) and by the choice of d, we have $N \cdot \|x\|_\infty \leq K_p \cdot \frac{2v_d}{d} \cdot N < \varrho$. In particular, $\sigma = \varrho$, and hence the above gives

$$\|I_{Y,Z}(x)\| \leq \varrho + M^{\frac{1}{q}} \cdot \varrho^{\frac{r}{q}} < \varepsilon$$

by the choice of ϱ.

The proof for $I_{Z_{q'},Y^*}$ is similar. One first needs to prove a dual version of Lemma 5, which is easier since in E_n we have an explicit formula for the norm, and then one needs to obtain a series of estimates as in the proof of Theorem 6. We first observe that Y^* is isomorphic to $W = (\bigoplus_{n=1}^\infty E_n)_{\ell_{p'}}$ by (2.4), and so it is sufficient to show that the formal inclusion map $I_{Z_{q'},W}$ is finitely strictly singular. So let us fix $\varepsilon > 0$, and then choose $\varrho > 0$ such that

$$\varrho + \varrho^{1-\frac{q'}{p'}} < \varepsilon.$$

We may and shall assume that $p < q \leq 2$. Indeed, given $p < q_1 < q_2$, we have

$$I_{Z_{q_2'},W} = I_{Z_{q_1'},W} \circ I_{Z_{q_2'},Z_{q_1'}}.$$

Now choose $n_0 \in \mathbb{N}$ such that

$$v_n < \varrho^{1-\frac{q'}{p'}} \quad \text{for all } n \geq n_0.$$

Set $N = (\sum_{n=1}^{n_0} n^{p'})^{\frac{1}{p'}}$ and choose $d \in \mathbb{N}$ with $d^{-\frac{1}{q'}} \cdot N < \varrho$.

Given a subspace H of $Z_{q'}$ of dimension at least d, use Milman's lemma again to find

$$x = \sum_{n=1}^{\infty}\sum_{j=1}^{n} x_{n,j} e_{2,j}^{(n)} \in H$$

with $\|x\|_{Z_{q'}} = 1$ such that

$$|x_{m,i}| = \|x\|_\infty = \sup_{n\in\mathbb{N},\, 1\leq j\leq n} |x_{n,j}|$$

for at least d pairs (m,i). Since $2 \leq q'$, we have

$$1 = \|x\|_{Z_{q'}} \geq \|x\|_{\ell_{q'}} \geq \|x\|_\infty \cdot d^{\frac{1}{q'}}, \quad \text{and so} \quad \|x\|_\infty \leq d^{-\frac{1}{q'}} < \frac{\varrho}{N} \leq \varrho. \tag{4.6}$$

Now fix $n \in \mathbb{N}$ with $n \geq n_0$. On the one hand, we have

$$\sum_{j=1}^{n} |x_{n,j}|^{p'} = \sum_{j=1}^{n} |x_{n,j}|^{p'-q'} |x_{n,j}|^{q'} \leq \varrho^{p'-q'} \cdot \left(\sum_{j=1}^{n}|x_{n,j}|^2\right)^{\frac{q'}{2}},$$

where we used $\|x\|_\infty < \varrho$ and that $2 \leq q'$. On the other hand, by the choice of n_0, and since $q' < p'$, we have

$$v_n^{p'}\left(\sum_{j=1}^{n}|x_{n,j}|^2\right)^{\frac{p'}{2}} \leq \varrho^{p'-q'} \cdot \left(\sum_{j=1}^{n}|x_{n,j}|^2\right)^{\frac{q'}{2}}.$$

The previous two inequalities imply that

$$\Big\| \sum_{j=1}^{n} x_{n,j} e_j^{(n)} \Big\|_{E_n}^{p'} \leq \varrho^{p'-q'} \cdot \Big\| \sum_{j=1}^{n} x_{n,j} e_{2,j}^{(n)} \Big\|_{\ell_2^n}^{q'}.$$

We deduce the following estimates:

$$\begin{aligned}
\|I_{Z_{q'},W}(x)\| &= \bigg(\sum_{n=1}^{\infty} \Big\| \sum_{j=1}^{n} x_{n,j} e_j^{(n)} \Big\|_{E_n}^{p'} \bigg)^{\frac{1}{p'}} \\
&\leq \bigg(\sum_{n=1}^{n_0} \Big\| \sum_{j=1}^{n} x_{n,j} e_j^{(n)} \Big\|_{E_n}^{p'} \bigg)^{\frac{1}{p'}} + \varrho^{1-\frac{q'}{p'}} \cdot \bigg(\sum_{n>n_0} \Big\| \sum_{j=1}^{n} x_{n,j} e_{2,j}^{(n)} \Big\|_{\ell_2^n}^{q'} \bigg)^{\frac{1}{p'}} \\
&\leq \|x\|_\infty \cdot N + \varrho^{1-\frac{q'}{p'}} \cdot \|x\|_{Z_{q'}}^{\frac{q'}{p'}} \\
&\leq \varrho + \varrho^{1-\frac{q'}{p'}} < \varepsilon,
\end{aligned}$$

where we recall that $N = \big(\sum_{n=1}^{n_0} n^{p'}\big)^{\frac{1}{p'}}$, and we used (4.6), the choice of d, and the choice of ϱ. □

We are now ready to prove our main result.

Proof of Theorem A. We first consider the case $1 < p < 2$ and $p < q < \infty$. Put

$$\eta = \frac{1}{p} - \frac{1}{2},$$

and define $f\colon \mathbb{N} \to \mathbb{R}$ by setting $f(n) = n^{-\eta}$ for each $n \in \mathbb{N}$. For an infinite set $M \subset \mathbb{N}$ we define a decreasing sequence $\boldsymbol{w}_M = (\boldsymbol{w}_M(n))_{n=1}^{\infty}$ in $(0, 1]$ as follows. Let $m_1 < m_2 < \cdots$ be the elements of M. We set $\boldsymbol{w}_M(1) = 1$ and $\boldsymbol{w}_M(2^{3^{m_k}}) = f(2^k)$ for each $k \in \mathbb{N}$, and then extend the definition of $\boldsymbol{w}_M$ to the rest of $\mathbb{N}$ by linear interpolation. It is clear that $\boldsymbol{w}_M(n) \geq n^{-\eta}$ for all $n \in \mathbb{N}$. We will show that for infinite sets $M \subset N \subset \mathbb{N}$ with $N \setminus M$ also infinite, the sequences $\boldsymbol{v} = \boldsymbol{w}_N$ and $\boldsymbol{w} = \boldsymbol{w}_M$ satisfy condition (4.1) of Theorem 6. Hence, by Corollary 7 we will have $\mathcal{J}^{I_{Y_{\boldsymbol{v}},Z_q}} \subsetneq \mathcal{J}^{I_{Y_{\boldsymbol{w}},Z_q}}$. Let us first explain how we complete the proof of our main theorem from here. We fix a chain $\mathcal{C}$ of size the continuum consisting of infinite subsets of $\mathbb{N}$ with any two having infinite difference. For $M \in \mathcal{C}$ put $Y_M = Y_{p,\boldsymbol{w}_M}$. As shown above, the closed ideals $\mathcal{J}^{I_{Y_M,Z_q}}$, $M \in \mathcal{C}$, are pairwise distinct and comparable. Moreover, for each $M \in \mathcal{C}$, the operator I_{Y_M,Z_q} is finitely strictly singular by Proposition 8, and it is clearly not compact. Hence the ideal $\mathcal{J}^{I_{Y_M,Z_q}}$ lies between $\mathcal{J}^{I_{p,q}}$ and $\mathcal{FS}$.

Using the same p and q, we now consider ideals in $\mathcal{L}(\ell_{q'}, \ell_{p'})$. For each $M \in \mathcal{C}$, we have $I_{Z_{q'},Y_M^*} = I_{Y_M,Z_q}^*$, and hence by simple duality we have

$$\mathcal{J}^{I_{Z_{q'},Y_M^*}}(\ell_{q'}, \ell_{p'}) = \big\{T^* : T \in \mathcal{J}^{I_{Y_M,Z_q}}\big\}.$$

Thus $\{\mathcal{J}^{I_{Z_{q'},Y_M^*}} : M \in \mathcal{C}\}$ is a chain of closed ideals in $\mathcal{L}(\ell_{q'}, \ell_{p'})$ of size the continuum. By Proposition 8, the operators $I_{Z_{q'},Y_M^*}$, $M \in \mathcal{C}$, are finitely strictly singular, and clearly not compact, so these ideals also lie between $\mathcal{J}^{I_{q',p'}}$ and $\mathcal{FS}$. Since $q' < p'$ and $2 < p'$, we have covered all remaining cases.

Let us now return to our claim: for infinite sets $M \subset N \subset \mathbb{N}$ with $N \setminus M$ also infinite, the sequences $\boldsymbol{v} = \boldsymbol{w}_N$ and $\boldsymbol{w} = \boldsymbol{w}_M$ satisfy condition (4.1) of Theorem 6. Let $m_1 < m_2 < \cdots$ and $n_1 < n_2 < \cdots$ be the elements of M and N, respectively. Fix $l \in \mathbb{N}$. We will show that for all sufficiently large $n \in \mathbb{N}$, we have

$$\frac{\boldsymbol{w}_N\big(n^{\frac{1}{3}}\big)}{\boldsymbol{w}_M(n)} \leq 2^{-\eta l},$$

which proves the claim. Since $N \setminus M$ is infinite, there exists $k_0 \in \mathbb{N}$ such that for all $n \geq m_{k_0}$ we have

$$|N \cap \{1, \dots, n\}| \geq |M \cap \{1, \dots, n\}| + l + 2.$$

Fix $n > 2^{3^{m_{k_0+1}}}$. This defines $k > k_0$ such that $2^{3^{m_k}} \leq n < 2^{3^{m_{k+1}}}$. It follows that

$$\boldsymbol{w}_M(n) \geq \boldsymbol{w}_M\big(2^{3^{m_{k+1}}}\big) = f\big(2^{k+1}\big). \tag{4.7}$$

Next define $k' \in \mathbb{N}$ by $n_{k'} \leq m_k - 1 < n_{k'+1}$. By the choice of k_0, and since $m_k - 1 \geq m_{k_0}$, we have

$$k' = |N \cap \{1, \dots, m_k - 1\}| \geq |M \cap \{1, \dots, m_k - 1\}| + l + 2 = k + l + 1.$$

It follows that

$$\boldsymbol{w}_N\big(n^{\frac{1}{3}}\big) \leq \boldsymbol{w}_N\big(2^{3^{m_k - 1}}\big) \leq \boldsymbol{w}_N\big(2^{3^{n_{k'}}}\big) = f\big(2^{k'}\big) \leq f\big(2^{k+l+1}\big). \tag{4.8}$$

Putting together (4.7) and (4.8), we obtain

$$\frac{\boldsymbol{w}_N\big(n^{\frac{1}{3}}\big)}{\boldsymbol{w}_M(n)} \leq \frac{f\big(2^{k+l+1}\big)}{f\big(2^{k+1}\big)} = 2^{-\eta l}.$$

This holds for any $n > 2^{3^{m_{k_0+1}}}$, so the proof of our claim is complete. □

We conclude this section with a proof of Theorem B. This will be very similar to the general case but simpler. We shall still rely on our key lemma from Section 3. From now on we fix $1 < p < 2 < q < \infty$. As usual, for a decreasing sequence $\boldsymbol{v} = (v_n)$ in $(0, 1]$ we consider the complemented subspace $Y_{\boldsymbol{v}} = Y_{p,\boldsymbol{v}}$ of ℓ_p with corresponding projection $P_{\boldsymbol{v}} = P_{p,\boldsymbol{v}}$ as introduced in Section 2.5. Since $2 < q$, the formal inclusion

$$I_{Z_q,q} \colon Z_q = \Big(\bigoplus_{n=1}^{\infty} \ell_2^n\Big)_{\ell_q} \to \ell_q = \Big(\bigoplus_{n=1}^{\infty} \ell_q^n\Big)_{\ell_q}$$

given by

$$I_{Z_q,q}\big(e_{2,j}^{(n)}\big) = e_{q,j}^{(n)}$$

is bounded, and hence, so is the formal inclusion $I_{Y_{\boldsymbol{v}},q} = I_{Z_q,q} \circ I_{Y_{\boldsymbol{v}},Z_q}$. We shall consider the closed ideal $\mathcal{J}^{I_{Y_{\boldsymbol{v}},q}}$ which contains the operator $S_{\boldsymbol{v}} = I_{Y_{\boldsymbol{v}},q} \circ P_{\boldsymbol{v}}$.

As before, we first distinguish ideals corresponding to different sequences.

Theorem 9. *Let* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *be as in Theorem* 6. *Then* $S_{\boldsymbol{w}} \notin \mathcal{J}^{I_{Y_{\boldsymbol{v}},q}}$.

Proof. We follow closely the proof of Theorem 6. For each $n \in \mathbb{N}$ let

$$F_n = F_{p,v_n}^{(n)} \quad \text{and} \quad G_n = F_{p,w_n}^{(n)}$$

with bases $\{f_j^{(n)} : 1 \leq j \leq n\}$ and $\{g_j^{(n)} : 1 \leq j \leq n\}$, respectively. To simplify notation we write $Y = Y_{\boldsymbol{v}}$ and $S = S_{\boldsymbol{w}}$. Thus

$$I_{Y,q}: Y = \left(\bigoplus_{n=1}^{\infty} F_n \right)_{\ell_p} \to \ell_q = \left(\bigoplus_{n=1}^{\infty} \ell_q^n \right)_{\ell_q}$$

is given by

$$I_{Y,q}\big(f_j^{(n)}\big) = e_{q,j}^{(n)},$$

and

$$S: \ell_p = \left(\bigoplus_{m=1}^{\infty} \ell_p^{k_m} \right)_{\ell_p} \to \ell_q$$

is the composite

$$S = I_{Y_{\boldsymbol{w}},q} \circ P_{\boldsymbol{w}}.$$

Note that $S(g_i^{(m)}) = e_{q,i}^{(m)}$.

For each $m \in \mathbb{N}$ we define $\Phi_m \in \mathcal{L}(\ell_p, \ell_q)^*$ by setting

$$\Phi_m(V) = \frac{1}{m} \sum_{i=1}^{m} \big\langle e_{q',i}^{(m)}, V(g_i^{(m)}) \big\rangle, \quad V \in \mathcal{L}(\ell_p, \ell_q).$$

Since $\|\Phi_m\| \le 1$ for all m, the sequence (Φ_m) has a ω^*-accumulation point Φ in the unit ball of $\mathcal{L}(\ell_p, \ell_q)^*$. Note that $\Phi_m(S) = 1$ for all m, and hence $\Phi(S) = 1$. The proof will be complete if we can show that $\mathcal{J}^{I_{Y,q}}$ is contained in the kernel of Φ. To see this, fix $A \in \mathcal{L}(\ell_q)$ and $B \in \mathcal{L}(\ell_p, Y)$ with $\|A\| \le 1$ and $\|B\| \le 1$. It is sufficient to show that

$$\Phi_m(A I_{Y,q} B) \to 0 \quad \text{as } m \to \infty. \tag{4.9}$$

Let $B_m: G_m \to Y$ denote the restriction to G_m of B. Exactly as in Theorem 6 we use Lemma 4 to show that

$$\frac{1}{m} \sum_{i=1}^{m} \big\| B_m\big(g_i^{(m)}\big) \big\|_{\infty} \to 0 \quad \text{as } m \to \infty, \tag{4.10}$$

and obtain the estimate

$$|\Phi_m(A I_{Y,q} B)| \le \frac{1}{m} \sum_{i=1}^{m} \big\| I_{Y,q} B_m\big(g_i^{(m)}\big) \big\|_{\ell_q}. \tag{4.11}$$

At this point we depart from the proof of Theorem 6 as the argument becomes simpler. Let

$$x = \sum_{n=1}^{\infty} \sum_{j=1}^{n} x_{n,j} f_j^{(n)} \in Y$$

with $\|x\|_Y \le 1$. Then for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{j=1}^{n} |x_{n,j}|^q &= \sum_{j=1}^{n} |x_{n,j}|^{q-2} \cdot |x_{n,j}|^2 \\
&\le C_p^2 \cdot \|x\|_{\infty}^{q-2} \cdot \Bigg\| \sum_{j=1}^{n} x_{n,j} f_j^{(n)} \Bigg\|_{F_n}^2 \\
&\le C_p^2 \cdot \|x\|_{\infty}^{q-2} \cdot \Bigg\| \sum_{j=1}^{n} x_{n,j} f_j^{(n)} \Bigg\|_{F_n}^p .
\end{aligned}$$

It follows that

$$\begin{aligned}
\|I_{Y,q}(x)\|_{\ell_q}^q &= \sum_{n=1}^{\infty}\sum_{j=1}^{n}|x_{n,j}|^q \\
&\le C_p^2 \cdot \|x\|_\infty^{q-2} \cdot \sum_{n=1}^{\infty}\Bigg\|\sum_{j=1}^{n} x_{n,j} f_j^{(n)}\Bigg\|_{F_n}^p \\
&= C_p^2 \cdot \|x\|_\infty^{q-2} \cdot \|x\|_Y^p \\
&\le C_p^2 \cdot \|x\|_\infty^{q-2}.
\end{aligned}$$

Applying this to (4.11), we obtain

$$\begin{aligned}
|\Phi_m(AI_{Y,q}B)| &\le C_p^{\frac{2}{q}} \cdot \frac{1}{m}\sum_{i=1}^{m} \|B_m(g_i^{(m)})\|_\infty^{1-\frac{2}{q}} \\
&\le C_p^{\frac{2}{q}} \cdot \left(\frac{1}{m}\sum_{i=1}^{m} \|B_m(g_i^{(m)})\|_\infty\right)^{1-\frac{2}{q}},
\end{aligned}$$

using the fact that the function $t \mapsto t^{1-\frac{2}{q}}$ is concave. Now we use (4.10) to deduce (4.9), as required. □

Proof of Theorem B. The proof of Theorem A provides a continuum size chain $\mathcal{C}$ of infinite subsets of $\mathbb{N}$, and corresponding sequences $\boldsymbol{w}_M$, $M \in \mathcal{C}$. In turn, this leads to spaces $Y_M = Y_{p,\boldsymbol{w}_M}$ and closed ideals $\mathcal{J}^{I_{Y_M,q}}$. The proof of Theorem A shows that if $M, N \in \mathcal{C}$ and $M \subset N$, then $\boldsymbol{v} = \boldsymbol{w}_N$ and $\boldsymbol{w} = \boldsymbol{w}_M$ satisfy the hypotheses in Theorem 6. In particular, the unit vector basis of Y_N dominates the unit vector basis of Y_M, and so $\mathcal{J}^{I_{Y_N,q}} \subset \mathcal{J}^{I_{Y_M,q}}$. Moreover, by Theorem 9, this inclusion is strict. Thus, the family $\{\mathcal{J}^{I_{Y_M,q}} : M \in \mathcal{C}\}$ of closed ideals of $\mathcal{L}(\ell_p, \ell_q)$ is a chain and has size the continuum.

Finally, for each $M \in \mathcal{C}$, the operator $I_{Y_M,q}$ factors through the formal inclusion

$$I_{2,q}\colon \ell_2 = \left(\bigoplus_{n=1}^{\infty} \ell_2^n\right)_{\ell_2} \to \ell_q = \left(\bigoplus_{n=1}^{\infty} \ell_q^n\right)_{\ell_q}$$

via the formal inclusion $I_{Y_M,2}$. Thus $\mathcal{J}^{I_{Y_M,q}}$ is contained in $\mathcal{J}^{I_{2,q}}$, and it contains $\mathcal{J}^{I_{p,q}}$ since $I_{Y_M,q}$ is not compact. □

5. Open problems

There are a number of natural questions that remain or arise after our work. The first aim would be to answer Pietsch's question in the range $1 \le p < q$.

Problem 10. Given $1 \le p < q < \infty$, are there infinitely many closed ideals in $\mathcal{L}(\ell_1, \ell_q)$ or in $\mathcal{L}(\ell_p, c_0)$?

Even in the reflexive range we do not know the exact number of closed ideals.

Problem 11. Given $1 < p < q < \infty$, find the cardinality of the set of closed ideals in $\mathcal{L}(\ell_p, \ell_q)$.

We now know this is at least $\mathfrak{c}$. On the other hand, it is clear that the cardinality of $\mathcal{L}(\ell_p, \ell_q)$ is $\mathfrak{c}$, and hence there can be at most $2^{\mathfrak{c}}$ closed ideals. Of course, one could pose the same problem with ℓ_1 replacing ℓ_p, etc.

Note that in the case $1 < p < 2 < q < \infty$ we constructed two continuum chains of closed ideals. Are these equal? More generally, we could ask the following question about the lattice structure of closed ideals in $\mathcal{L}(\ell_p, \ell_q)$.

Problem 12. Do the closed ideals of $\mathcal{L}(\ell_p, \ell_q)$, after ignoring a finite number of them, form a chain?

So far all our new ideals are generated by a single operator. Note that if T is a non-zero compact operator, then $\mathcal{K} = \mathcal{J}^T$. It is then natural to ask the following.

Problem 13. Is $\mathcal{FS}$ generated by one operator? Are all closed ideals of $\mathcal{L}(\ell_p, \ell_q)$ generated by one operator?

Another candidate of a closed ideal, not representable by a single operator could be the closure of the union of one of the chains we defined in the previous section.

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